Multiplicative Calculus

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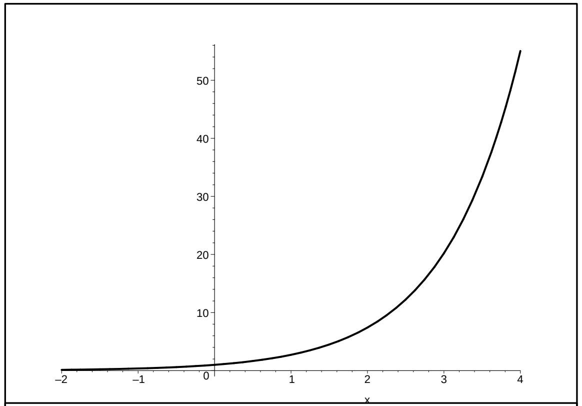
Motivation

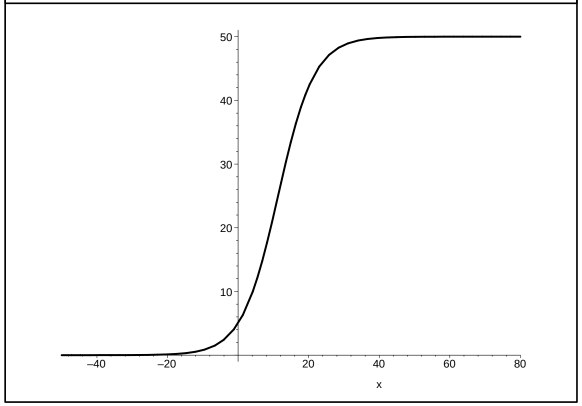
Solving simple differential equations:

$$y' = ky$$
$$y = Ce^{kt} \qquad \text{if } y > 0$$

Exponential Functions:
-constant growth rate
-unbounded growth



Logistic Growth Equation:
-bounded population growth
-decreasing growth rate



Motivation

Growth rate = Derivative:

$$y' = ky = Cke^{kt} \qquad \text{(not constant)}$$

The ratio $\dfrac{y'}{y} = k$ is constant.

But this is the *growth constant*, not the growth rate.

The growth rate is a multiplicative growth factor.

i.e. what you would multiply by to get the "next" function value.

A population that doubles each year has a growth rate of 2.
A population with a growth constant k has a growth rate of e^k.

Additive vs Multiplicative

ADDITIVE	MULTIPLICATIVE
additive slope	multiplicative slope
linear functions constant	exponential functions constant
$f(x) = mx + b$	$g(x) = Ca^x$
$f'(x) = m$	$g^*(x) = a$
$f(x+1) = f(x) + f'(x)$	$g(x+1) = g(x) \cdot g^*(x)$

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$f(x+1) = f(x) + f'(x)$	$g(x+1) = g(x) \cdot g^*(x)$
$\dfrac{f(x+h) - f(x)}{h}$	$\left(\dfrac{g(x+h)}{g(x)}\right)^{\frac{1}{h}}$
-addition	-multiplication
-subtraction	-division
-multiplication	-exponentiation

Multiplicative Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f^*(x) = \lim_{h \to 0} \left(\frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}}$$

Note: $f^*(x)$ is only defined where $f(x) \neq 0$.

Other Notation: $\frac{d^* f}{dx}$

Higher Order Derivatives: $f^{**}(x)$

n^{th} Derivative: $f^{*(n)}(x)$

Simplifying Formula

$$
\begin{aligned}
f^*(x) &= \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)}\right)^{\frac{1}{h}} \\
&= \lim_{h\to 0}\left(\frac{f(x+h)}{f(x)} - \frac{f(x)}{f(x)} + 1\right)^{\frac{1}{h}} \\
&= \lim_{h\to 0}\left(1 + \frac{f(x+h)-f(x)}{f(x)}\right)^{\frac{f(x)}{f(x+h)-f(x)}\cdot\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{f(x)}} \\
&= \lim_{h\to 0}\left[\left(1 + \frac{f(x+h)-f(x)}{f(x)}\right)^{\frac{f(x)}{f(x+h)-f(x)}}\right]^{\frac{f(x+h)-f(x)}{h}\cdot\frac{1}{f(x)}} \\
&= e^{\frac{f'(x)}{f(x)}} \\
&= e^{(\ln\circ|f|)'(x)}
\end{aligned}
$$

Differentiability

$$f^*(x) = e^{(\ln \circ |f|)'(x)}$$

Similarly, $\quad f^{**}(x) = e^{(\ln \circ f^*)'(x)} = e^{(\ln \circ |f|)''(x)}$

If $f(x) \neq 0$ and $f^{(n)}(x)$ exists, then $f^{*(n)}(x)$ exists and

$$f^{*(n)}(x) = e^{(\ln \circ |f|)^{(n)}(x)} \qquad \text{for } n = 0, 1, 2, \ldots$$

Note: For $n = 0 \quad f^{*(0)}(x) = e^{(\ln \circ |f|)^{(0)}(x)} = |f(x)|$

For $f : A \to \mathbb{R}$ non-zero:

f differentiable at x or on $A \Longrightarrow {}^*$differentiable at x or on A.

Differentiability

$$f^*(x) = e^{\frac{f'(x)}{f(x)}}$$

$$f'(x) = f(x) \cdot \ln\left(f^*(x)\right)$$

For $f : A \to \mathbb{R}$ non-zero:

$f\ ^*$differentiable at x or on $A \Longrightarrow$ differentiable at x or on A.

$f\ ^*$differentiable at x or on $A \Longleftrightarrow$ differentiable at x or on A.

Continuity

*Differentiability implies continuity

$$f^*(c) = \lim_{x \to c} \left(\frac{f(x)}{f(c)} \right)^{\frac{1}{x-c}}$$

We want $\lim\limits_{x \to c} f(x) - f(c) = 0$

For $f(c) \neq 0$:

$$\begin{aligned}
\lim_{x \to c} f(x) - f(c) = 0 \quad &\Longleftrightarrow \quad \lim_{x \to c} \left(\frac{f(x) - f(c)}{f(c)} \right) = 0 \\
&\Longleftrightarrow \quad \lim_{x \to c} \left(\frac{f(x)}{f(c)} - 1 \right) = 0 \\
&\Longleftrightarrow \quad \lim_{x \to c} \left(\frac{f(x)}{f(c)} \right) = 1
\end{aligned}$$

Continuity

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We want $\lim_{x \to c} \left(\frac{f(x)}{f(c)} \right) = 1$

$$\begin{aligned} \lim_{x \to c} \left(\frac{f(x)}{f(c)} \right) &= \lim_{x \to c} \left(\frac{f(x)}{f(c)} \right)^{\frac{1}{x-c} \cdot (x-c)} \\ &= \lim_{x \to c} \left[\left(\frac{f(x)}{f(c)} \right)^{\frac{1}{x-c}} \right]^{x-c} \\ &= [f^*(c)]^0 = 1 \end{aligned}$$

Continuity

Continuity does not imply *differentiability

$f(x) = |x| + 1$ at $x = 0$

$$\lim_{h \to 0^+} \left(\frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}} = \lim_{h \to 0^+} \left(\frac{|h| + 1}{1} \right)^{\frac{1}{h}}$$

$$= \lim_{h \to 0^+} (1 + h)^{\frac{1}{h}} = e$$

$$\lim_{h \to 0^-} \left(\frac{f(x+h)}{f(x)} \right)^{\frac{1}{h}} = \lim_{h \to 0^-} \left(\frac{|h| + 1}{1} \right)^{\frac{1}{h}}$$

$$= \lim_{h \to 0^-} (1 + |h|)^{\frac{1}{-|h|}} = e^{-1}$$

Constant Functions

If $f(x) = c \neq 0$ on (a, b), then

$$f^*(x) = e^{(\ln|c|)'} = e^0 = 1 \qquad \text{on } (a, b)$$

Conversely, if $f^*(x) = 1$ on (a, b), then

$$f^*(x) = e^{\frac{f'(x)}{f(x)}} = 1 \quad \text{implies } f(x) = c \neq 0$$

Constant functions:

additive derivative=0 (additive identity)

multiplicative derivative=1 (multiplicative identity)

Derivative Rules

$$\left(cf\right)^* (x) = f^*(x)$$

$$\left(fg\right)^* (x) = f^*(x) g^*(x) \quad \text{Product Rule}$$

$$\left(\frac{f}{g}\right)^* (x) = \frac{f^*(x)}{g^*(x)} \quad \text{Quotient Rule}$$

$$\left(f^g\right)^* (x) = f^*(x)^{g(x)} \cdot f(x)^{g'(x)}$$

$$\left(f \circ g\right)^* (x) = f^*\left(g(x)\right)^{g'(x)} \quad \text{Chain Rule}$$

$$\left(f + g\right)^* (x) = f^*(x)^{\frac{f(x)}{f(x)+g(x)}} \cdot g^*(x)^{\frac{g(x)}{f(x)+g(x)}} \quad \text{Sum Rule}$$

Examples

$f(x)$	$f^*(x)$
C	1
Ce^{kx}	e^k
Ca^x	a
Cx	$e^{\frac{1}{x}}$
$mx+b$	$e^{\frac{m}{mx+b}}$
Cx^n	$e^{\frac{n}{x}}$
$C\ln(x)$	$e^{\frac{1}{x\ln(x)}}$
$C\ln(g(x))$	$[g^*(x)]^{\frac{1}{\ln(g(x))}}$
$C\sin(x)$	$e^{\cot(x)}$
$C\cos(x)$	$e^{\tan(x)}$
$Ce^{\sin(x)}$	$e^{\cos(x)}$

Mean Value Theorem

If $f(x)$ is continuous on $[a,b]$ and *differentiable on (a,b), then there exists $a < c < b$ s.t.

$$f^*(c) = \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{b-a}}$$

This follows from the Mean Value Theorem applied to $(\ln \circ |f|)(x)$

Monotonicity

$f : (a, b) \to \mathbb{R}^{*}$differentiable.

If $f^*(x) > 1$ on (a, b), then f is strictly increasing.

If $f^*(x) < 1$ on (a, b), then f is strictly decreasing.

If $f^*(x) \geq 1$ on (a, b), then f is increasing.

If $f^*(x) \leq 1$ on (a, b), then f is decreasing.

Relative Extrema

$f : (a, b) \to \mathbb{R}$ twice *differentiable.

If f has a local extremum at $c \in (a, b)$, then $f^*(c) = 1$.

If $f^*(c) = 1$ and $f^{**}(c) > 1$, then f has a local minimum at c.

If $f^*(c) = 1$ and $f^{**}(c) < 1$, then f has a local maximum at c.

Approximation

$f'(c) =$ slope of tangent line at $x = c$
$f^*(c) =$ base of tangent exponential curve at $x = c$

Linear Approx: $L(x) = f(c) + f'(c)(x - c)$
Exponential Approx: $E(x) = f(c) \cdot f^*(c)^{x-c}$

Note:

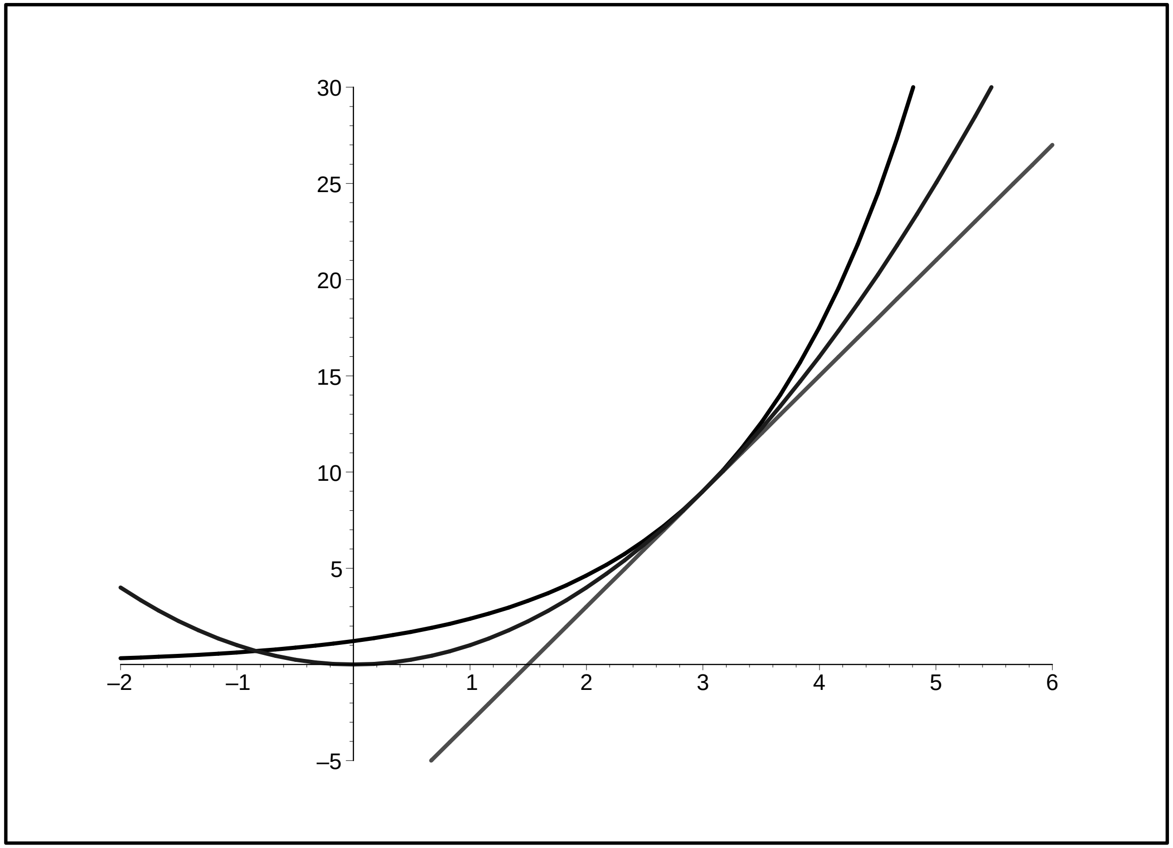
$$L(c) = f(c) \qquad L'(c) = f'(c) \qquad L^*(c) = f^*(c)$$

$$E(c) = f(c) \qquad E'(c) = f'(c) \qquad E^*(c) = f^*(c)$$

Approximation

Example 1: $f(x) = x^2$ at $c = 3$

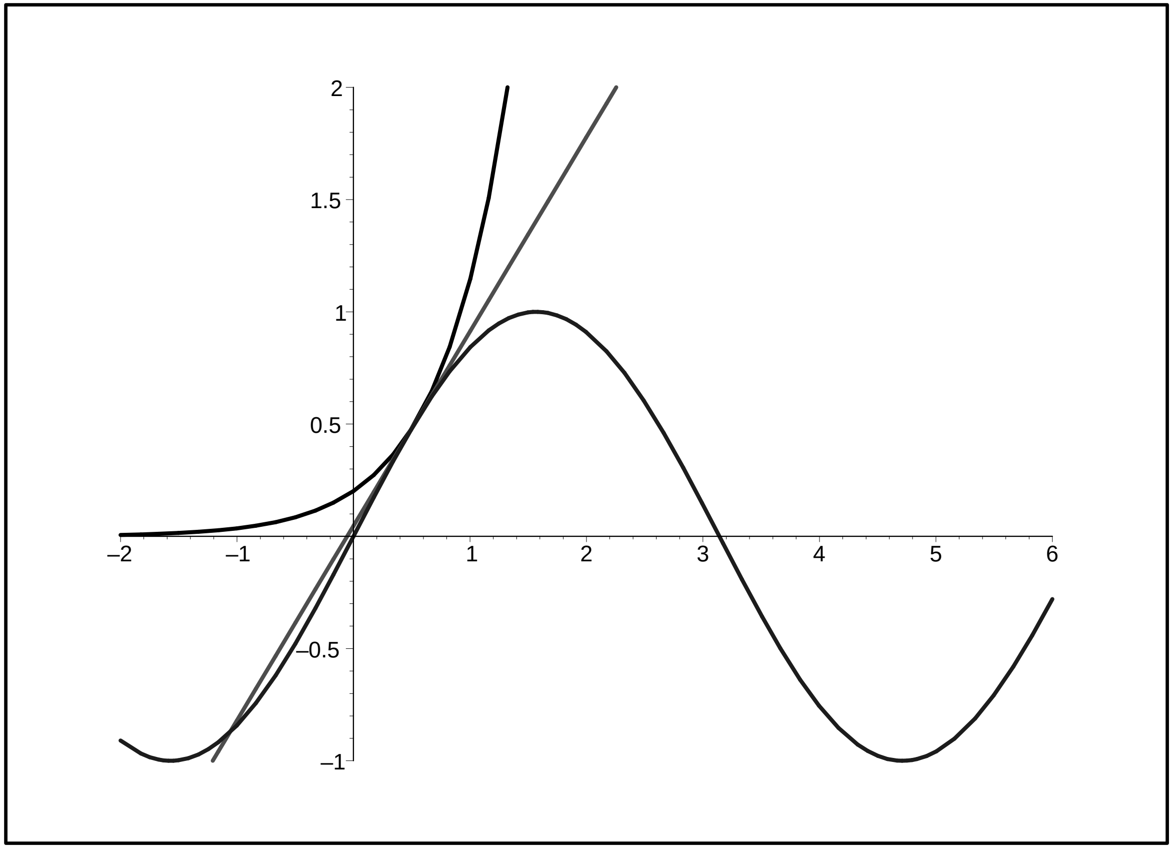
$$L(x) = 9 + 6(x - 3) \qquad E(x) = 9 \cdot \left(e^{\frac{2}{3}}\right)^{x-3}$$



Approximation

Example 2: $f(x) = \sin(x)$ at $c = \dfrac{\pi}{6}$

$$L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) \qquad E(x) = \frac{1}{2} \cdot \left(e^{\frac{\sqrt{3}}{2}}\right)^{x - \frac{\pi}{6}}$$



Multiplicative Integrals

Let $\mathcal{P}$ be a partition of $[a,b]$.

Riemann Sums: $$S(f,\mathcal{P}) = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

* Products: $$P(f,\mathcal{P}) = \prod_{i=1}^{n} |f(c_i)|^{(x_i - x_{i-1})}$$

If this product converges, we say f is $^*integrable$ and denote the limit by $\int_a^b f(x)^{dx}$

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$$\int_a^a f(x)^{dx} =$$

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$$\int_a^a f(x)^{dx} = 1 \qquad \text{and} \qquad \int_b^a f(x)^{dx} = \left(\int_a^b f(x)^{dx}\right)^{-1}$$

Antiderivatives

$$\int 1^{dx} = C \qquad \int k^{dx} = Ck^x \text{ for } k > 0$$

$$\int \left[e^{kx}\right]^{dx} = Ce^{\frac{kx^2}{2}} \qquad \int \left[e^{kx^n}\right]^{dx} = Ce^{\frac{kx^{n+1}}{n+1}}$$

$$\int \left[e^{\frac{k}{x}}\right]^{dx} = Cx^k \qquad \int \left[e^{\cos(x)}\right]^{dx} = Ce^{\sin(x)}$$

Simplifying Formula

If f is positive and Riemann integrable on $[a,b]$, then f is *integrable and

$$\int_a^b f(x)^{dx} = e^{\int_a^b (\ln \circ |f|)(x)dx}$$

This follows from

$$P(f,\mathcal{P}) = e^{\sum_{i=1}^n (\ln \circ |f|)(c_i)(x_i - x_{i-1})} = e^{S(\ln \circ |f|, \mathcal{P})}$$

Conversely, if f is Riemann integrable on $[a,b]$ then

$$\int_a^b f(x)dx = \ln\left(\int_a^b \left(e^{f(x)}\right)^{dx}\right)$$

Integration Rules

$$\int_a^b (f(x)^p)^{dx} = \left(\int_a^b f(x)^{dx}\right)^p$$

$$\int_a^b (f(x)g(x))^{dx} = \left(\int_a^b f(x)^{dx}\right)\left(\int_a^b g(x)^{dx}\right)$$

$$\int_a^b \left(\frac{f(x)}{g(x)}\right)^{dx} = \frac{\int_a^b f(x)^{dx}}{\int_a^b g(x)^{dx}}$$

$$\int_a^b f(x)^{dx} = \int_a^c f(x)^{dx} \cdot \int_c^b f(x)^{dx}$$

Fundamental Theorem

If $f : [a,b] \to \mathbb{R}$ is *differentiable and f^* is *integrable

$$\int_a^b f^*(x)^{dx} = \frac{f(b)}{f(a)}$$

Let $f : [a,b] \to \mathbb{R}$ be *integrable and $F(x) = \int_a^x f(t)^{dt}$

If f is continuous at $x \in [a,b]$, then F is *differentiable at x and $F^*(x) = f(x)$.

Taylor Products

Taylor Series:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Taylor Product:

$$f(x) = \prod_{k=0}^{n} \left[f^{*(k)}(a)\right]^{\frac{(x-a)^k}{k!}} \cdot \left[f^{*(n+1)}(c)\right]^{\frac{(x-a)^{n+1}}{(n+1)!}}$$

Remainder terms go to 0 and 1 respectively as $n \to \infty$

Taylor Products

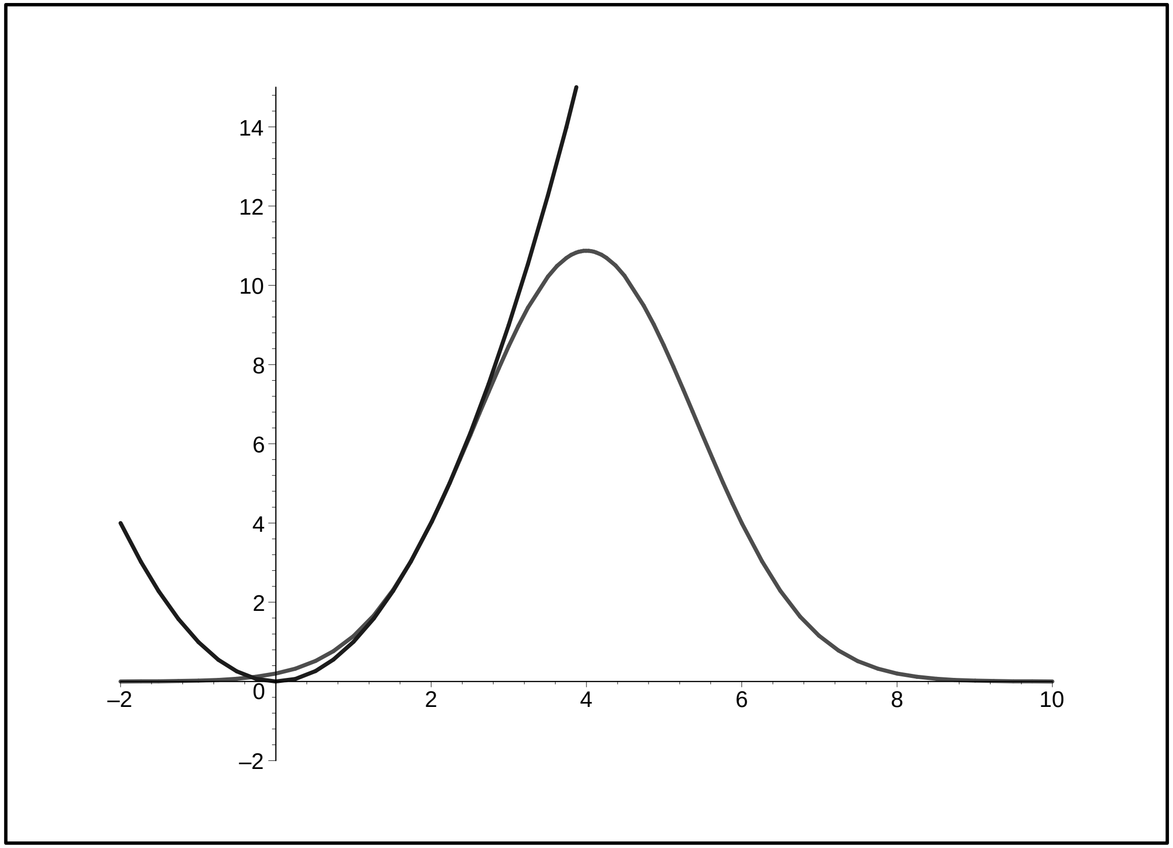
2^{nd} Order Approximation

$$
\begin{aligned}
E_2(x) &= \prod_{k=0}^{2} \left[f^{*(k)}(a) \right]^{\frac{(x-a)^k}{k!}} \\
&= f(a) \cdot [f^*(a)]^{x-a} \cdot [f^{**}(a)]^{\frac{(x-a)^2}{2}}
\end{aligned}
$$

Taylor Products

Example 1: $f(x) = x^2$ at $a = 2$

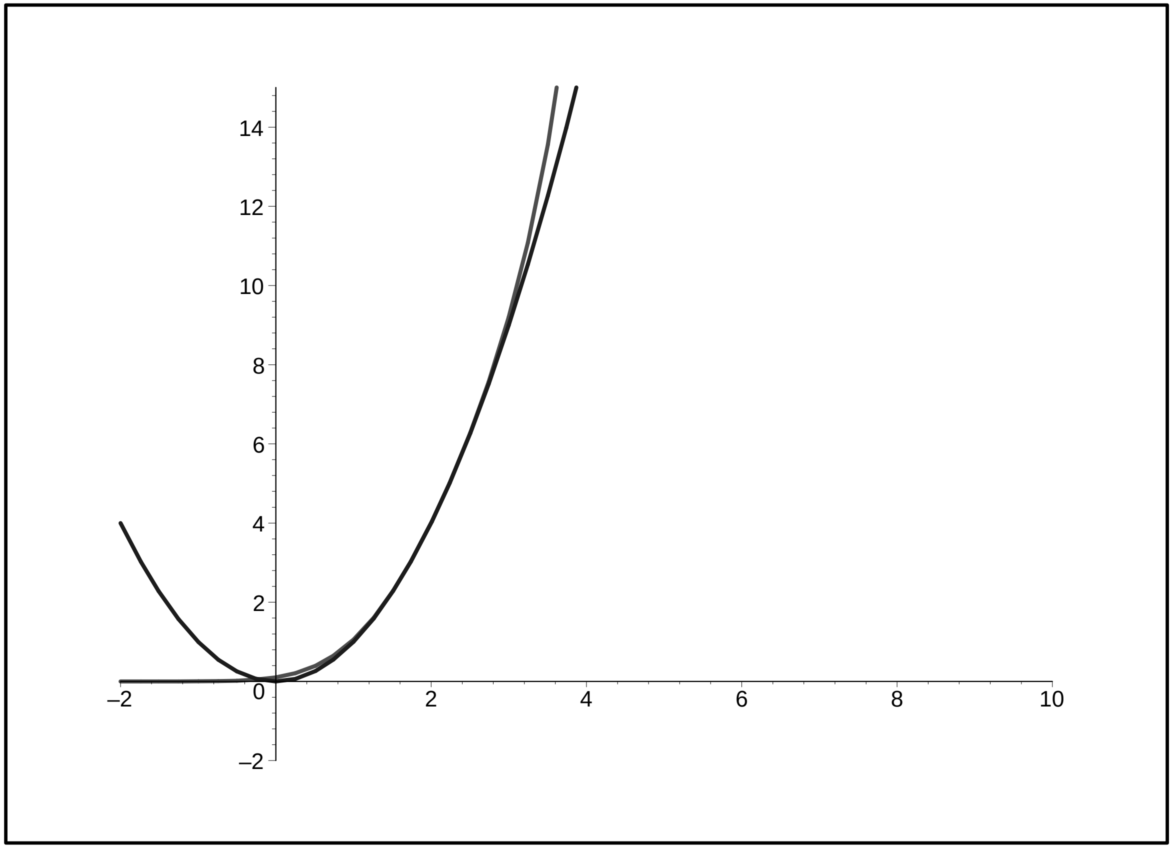
$$E_2(x) = 4 \cdot e^{x-2} \cdot e^{\frac{-1}{2} \cdot \frac{(x-2)^2}{2}}$$



Taylor Products

Example 1: $f(x) = x^2$ at $a = 2$

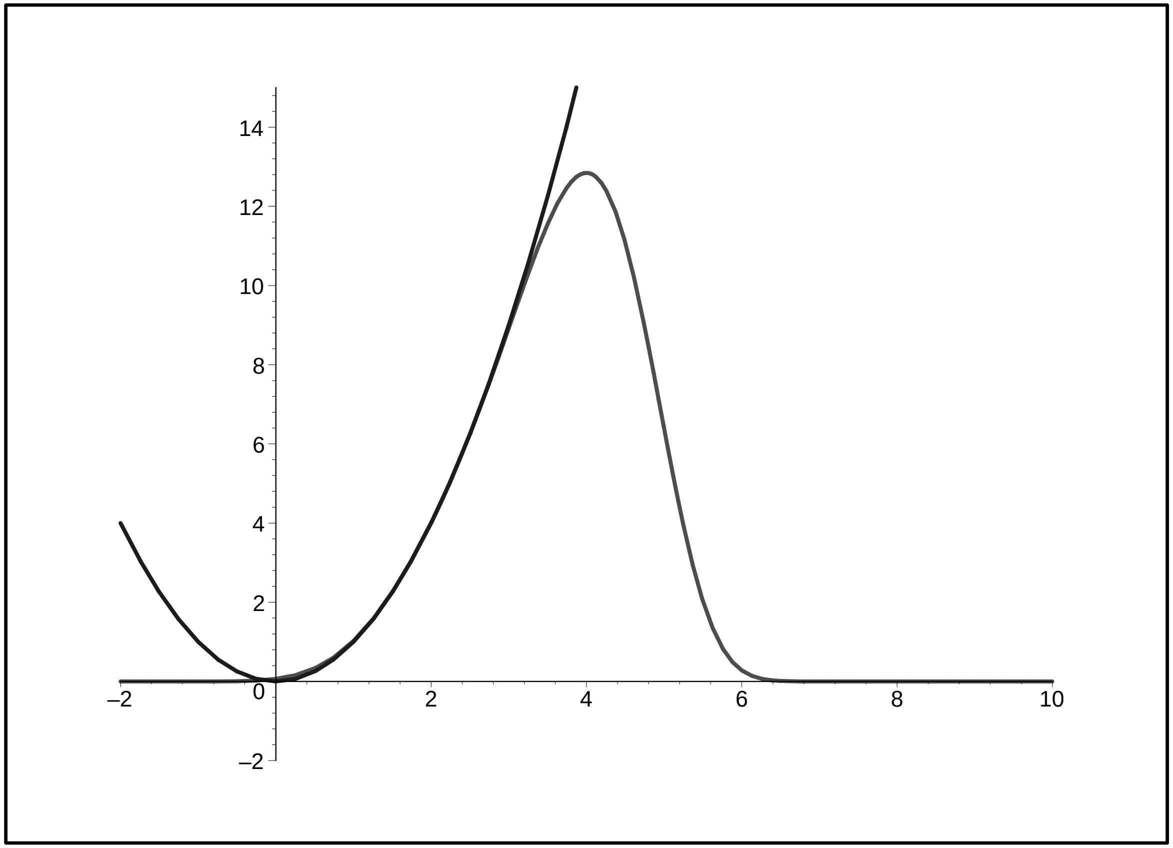
$$E_3(x) = E_2(x) \cdot \left[e^{\frac{1}{2}} \right]^{\frac{(x-2)^3}{6}}$$



Taylor Products

Example 1: $f(x) = x^2$ at $a = 2$

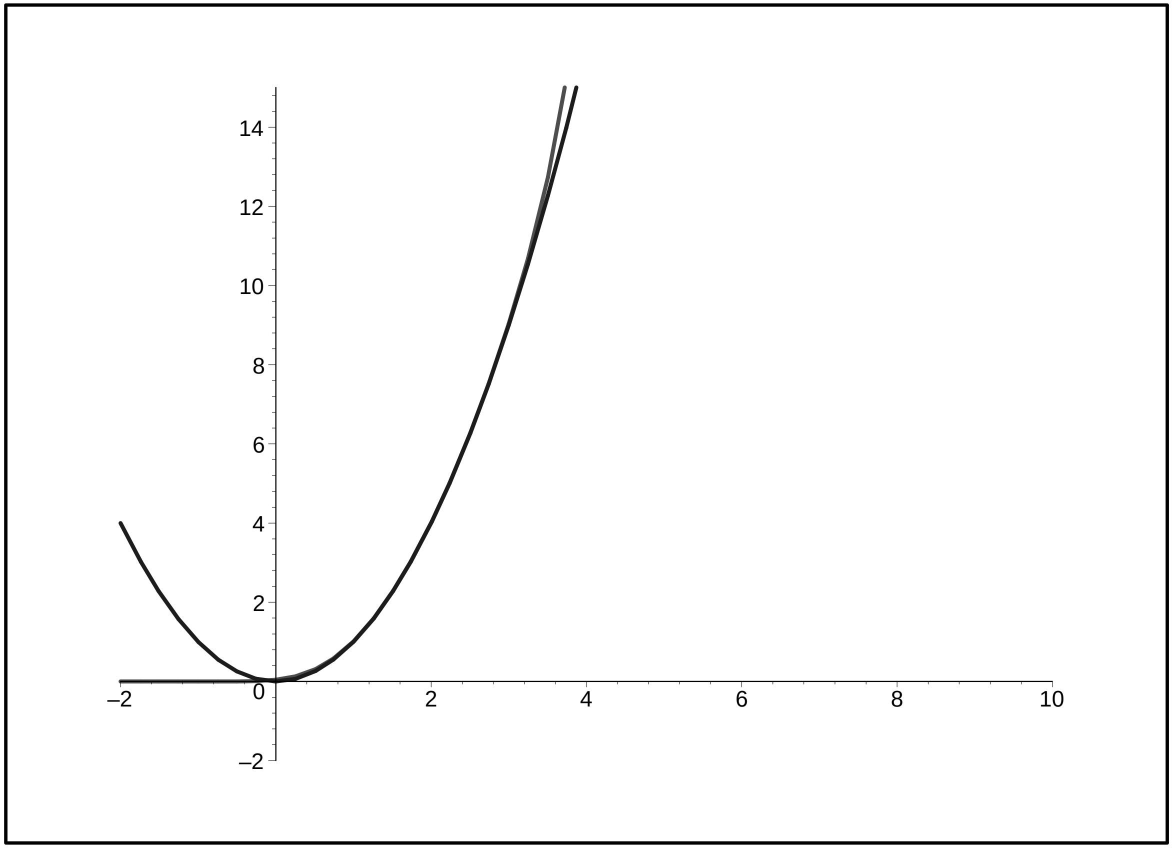
$$E_4(x) = E_3(x) \cdot \left[e^{\frac{-3}{4}} \right]^{\frac{(x-2)^4}{24}}$$



Taylor Products

Example 1: $f(x) = x^2$ at $a = 2$

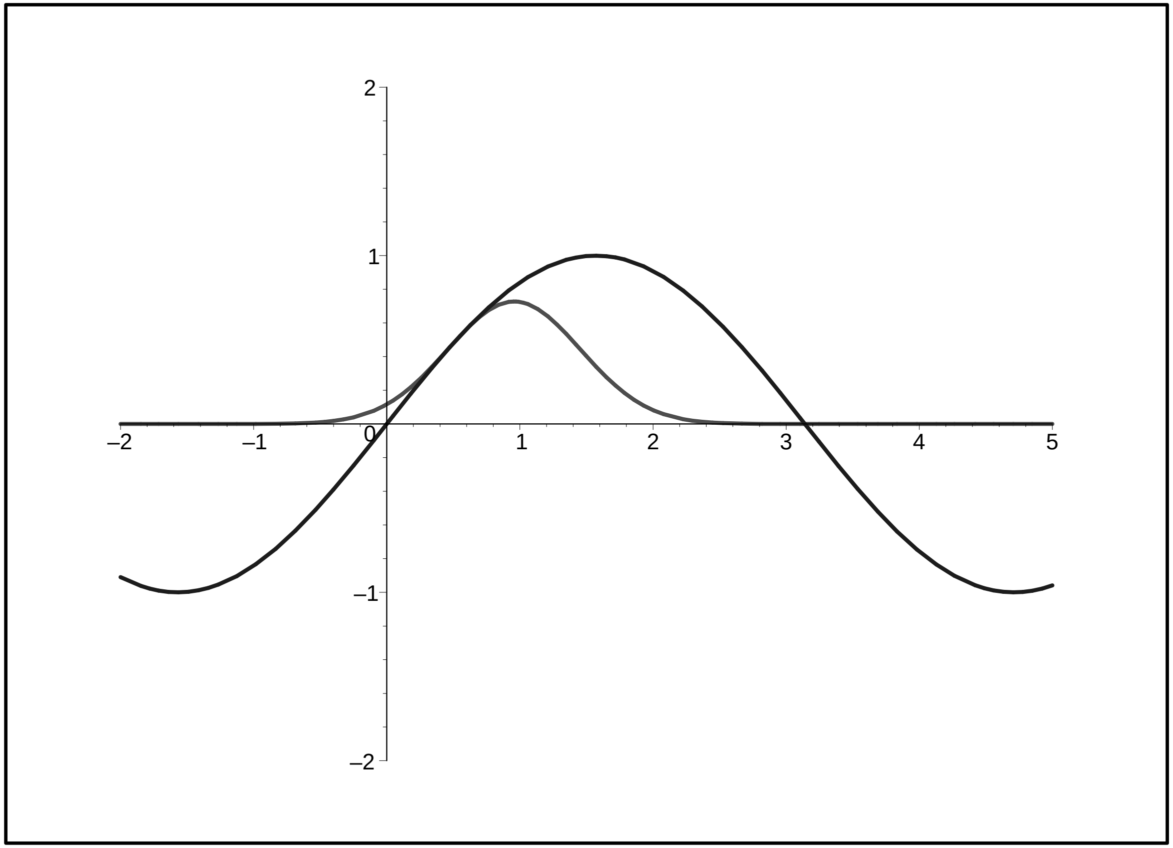
$$E_5(x) = E_4(x) \cdot \left[e^{\frac{3}{2}} \right]^{\frac{(x-2)^5}{120}}$$



Taylor Products

Example 2: $f(x) = \sin(x)$ at $a = \dfrac{\pi}{6}$

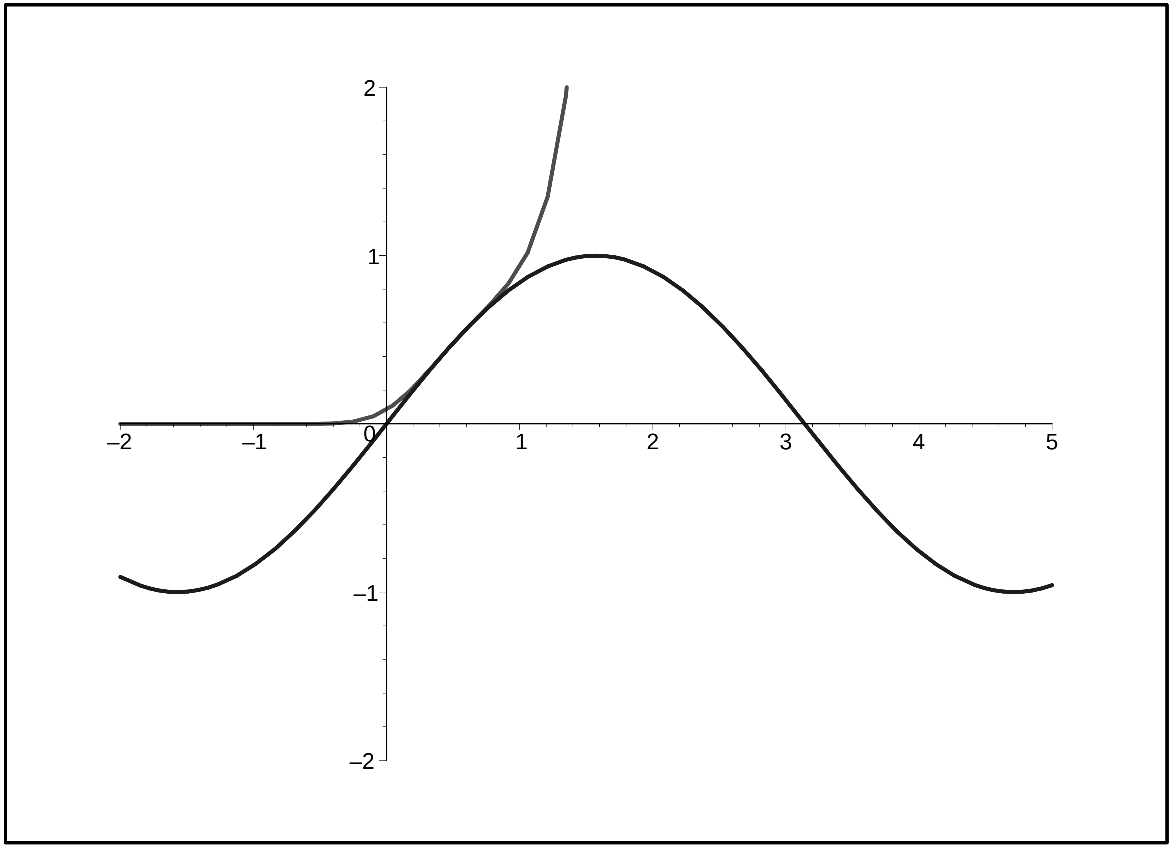
$$E_2(x) = \frac{1}{2} \cdot e^{\sqrt{3}\left(x-\frac{\pi}{6}\right)} \cdot e^{\frac{-4\left(x-\frac{\pi}{6}\right)^2}{2}}$$



Taylor Products

Example 2: $f(x) = \sin(x)$ at $a = \dfrac{\pi}{6}$

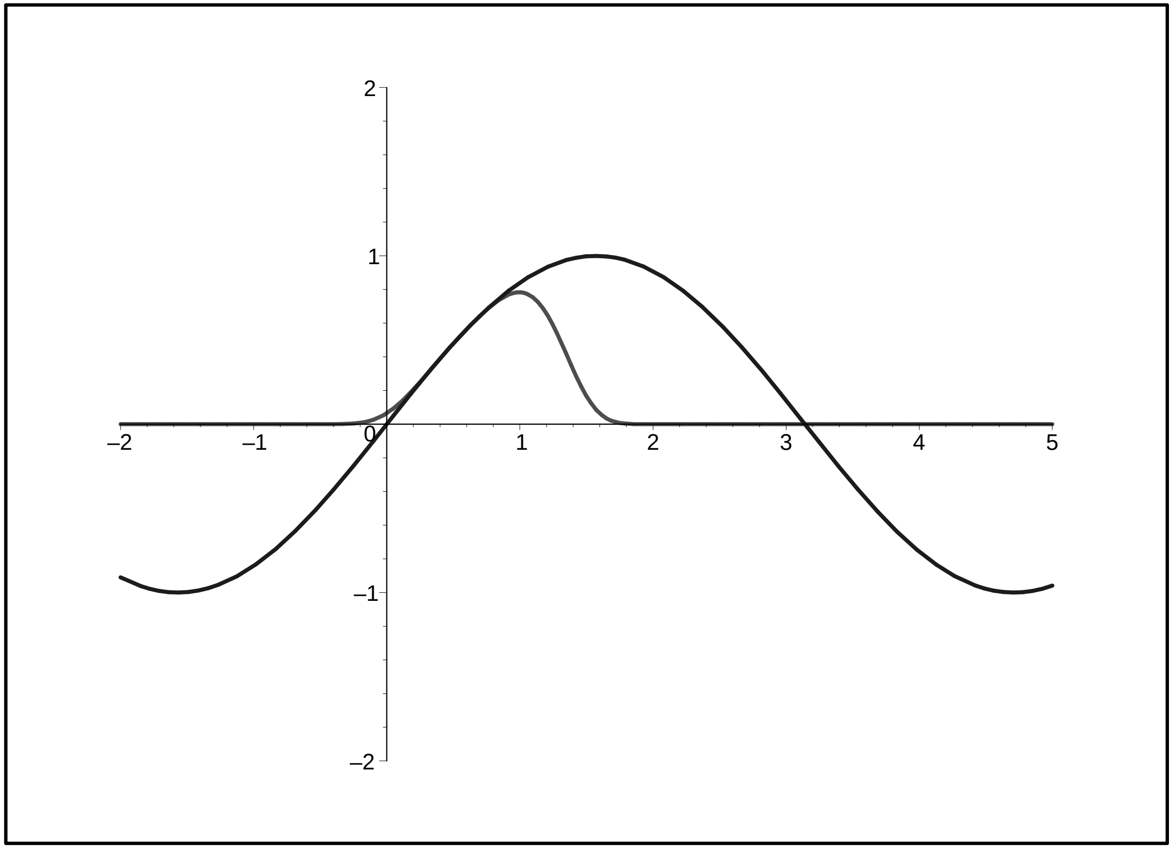
$$E_3(x) = E_2(x) \cdot \left[e^{8\sqrt{3}}\right]^{\frac{\left(x-\frac{\pi}{6}\right)^3}{6}}$$



Taylor Products

Example 2: $f(x) = \sin(x)$ at $a = \dfrac{\pi}{6}$

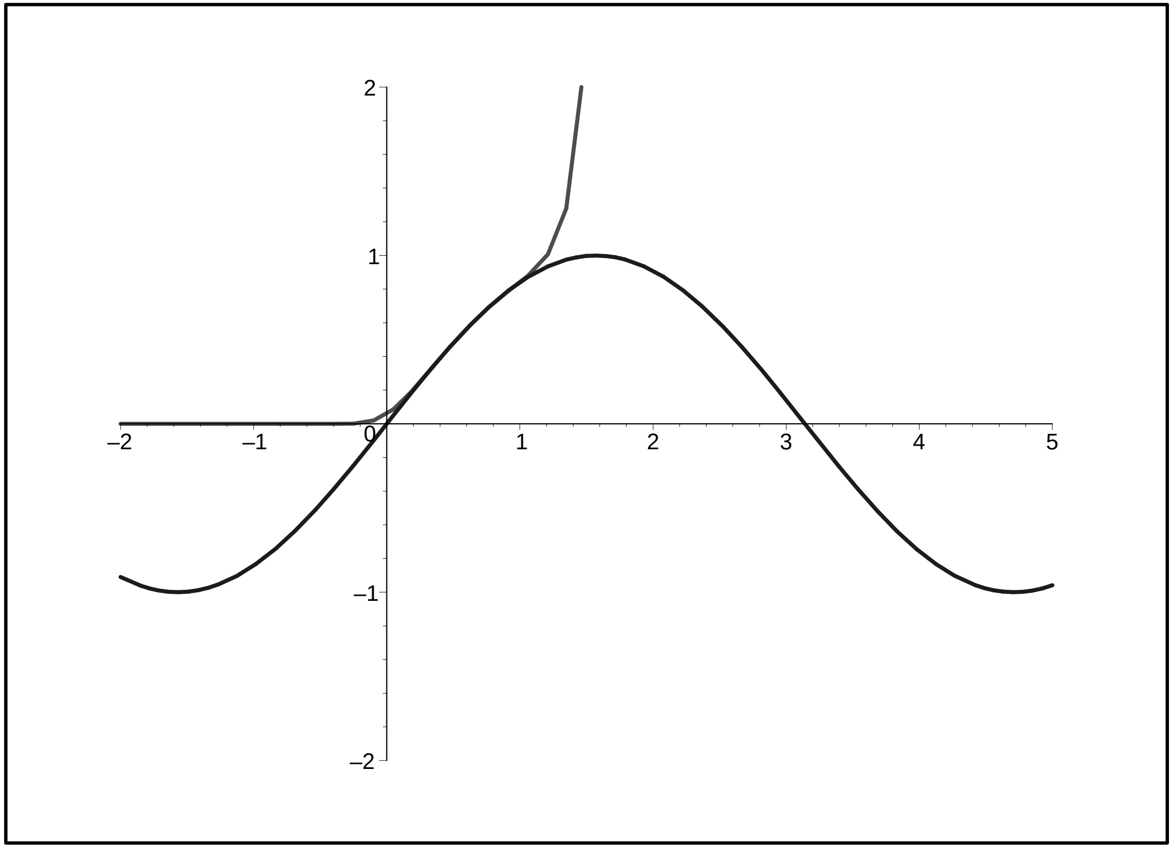
$$E_4(x) = E_3(x) \cdot \left[e^{-80}\right]^{\frac{\left(x-\frac{\pi}{6}\right)^4}{24}}$$



Taylor Products

Example 2: $f(x) = \sin(x)$ at $a = \dfrac{\pi}{6}$

$$E_5(x) = E_4(x) \cdot \left[e^{352\sqrt{3}}\right]^{\frac{\left(x-\frac{\pi}{6}\right)^5}{120}}$$



Other Calculi

If φ is a bijective function, define †derivative and †integral by

$$f^{\dagger}(x) = \varphi\left(\varphi^{-1} \circ f\right)'(x)$$

$$\int_a^b f(x) d^{\dagger} x = \varphi\left(\int_a^b \left(\varphi^{-1} \circ f\right)(x) dx\right)$$

Other Calculi

If φ is a bijective function, define †derivative and †integral by

$$f^\dagger(x) = \varphi\left(\varphi^{-1} \circ f\right)'(x)$$

$$f^*(x) = e^{(\ln \circ |f|)'(x)}$$

$$\int_a^b f(x) d^\dagger x = \varphi\left(\int_a^b \left(\varphi^{-1} \circ f\right)(x) dx\right)$$

$$\int_a^b f(x)^{dx} = e^{\int_a^b (\ln \circ |f|)(x) dx}$$

Applications

- Support for Newtonian Calculus
- Semigroups of linear operators
- Multiplicative metric spaces
- Multiplicative differential equations
- Multiplicative Calculus of variations
- Student projects

Multiplicative Metric Spaces

Define multiplicative absolute value for $x \in \mathbb{R}^+$

$$|x|^* = \begin{cases} x & \text{if } x \geq 1 \\ \frac{1}{x} & \text{if } x < 1 \end{cases}$$

Define the multiplicative distance for $x, y \in \mathbb{R}^+$

$$d^*(x, y) = \left| \frac{x}{y} \right|^*$$

Multiplicative Metric Spaces

Define the multiplicative distance for $x, y \in \mathbb{R}^+$

$$d^*(x, y) = \left|\frac{x}{y}\right|^*$$

Properties:

1. $d^*(x, y) \geq 1 \quad \forall x, y \in \mathbb{R}^+$
2. $d^*(x, y) = 1 \Longleftrightarrow x = y$
3. $d^*(x, y) = d^*(y, x) \quad \forall x, y \in \mathbb{R}^+$
4. $d^*(x, z) \leq d^*(x, y) d^*(y, z) \quad \forall x, y, z \in \mathbb{R}^+$

Multiplicative Metric Spaces

Define the multiplicative distance for $x, y \in \mathbb{R}^+$

$$d^*(x, y) = \left|\frac{x}{y}\right|^*$$

Multiplicative convergence in $\mathbb{R}^+$

$$(x_n)_{n=1}^{\infty} \xrightarrow{*} x \iff \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad s.t. \quad d^*(x_n, x) < 1 + \varepsilon \quad \forall n > N$$

$$\iff \forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad s.t. \quad \left|\frac{x_n}{x}\right|^* < 1 + \varepsilon \quad \forall n > N$$

Multiplicative Metric Spaces

A matrix A is positive if $\mathbf{x}^T A\mathbf{x} > 0$ for every $n-$vector $\mathbf{x}$

$\mathbb{M}_n^+ =$ set of positive $(n \times n)-$matrices

$A \in \mathbb{M}_n^+$ then its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$

Define multiplicative norm of A

$$\|A\|^* = \prod_{i=1}^{n} |\lambda_i|^*$$

Define multiplicative distance for $A, B \in \mathbb{M}_n^+$

$$d^*(A, B) = \left\|AB^{-1}\right\|^*$$

References

Agamirza E. Bashirov, Emine Misirli Kurpmar, Ali Ozyapici, *Multiplicative Calculus and its applications*, J. Math. Annal. Appl. 337 (2008) 36-48.

Duff Campbell, *Multiplicative Calculus and Student Projects*, Primus IX (4)(1999) 327-333.

Michael Grossman, Robert Katz, Non-Newtonian Calculus, Lee Press, Pigeon Cove, MA, 1972.

Dick Stanley, *A Multiplicative Calculus*, Primus IX (4)(1999) 310-326.