Multiply it!

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Introduction. A long time ago when I was a student I wanted to look how far you could come with generalizing the complex multiplication to higher dimensions. The first two or three months I did not make much progress because I constantly tried to lay isomorphisms between n dimensional space and the complex plane.
Only when I realized that the complex multiplication is nothing special (later we will see it has a parameter $T = -1$ when we parameterize multiplications) serious progress did set in.
When I write the complex plane is 'nothing special' this should not be taken as a negative thing, it simply means there is a whole lot more out there that is at least as beautiful as our well known complex plane...
This article should not be viewed as a comprehensive treatment of the general n dimensional situation, it is only a step up so it could be used in creating fractals. So in theory we would only need functions like $f(X) = X^2 + c$ in the n dimensional case but with the same easy we take higher polynomials but also well known functions like exp, log, sin and cos and so on and so on.

As usual: Some things about notation or 'how to write it down'. In n dimensional real space we use the natural coordinate system with unit basis vectors.
The first axis is always the real number system (the real number line). Since we know the real line is closed under addition and multiplication it is handy not to start counting the basis vectors with e_1 but with e_0
Therefore the basis we use is given by n basis vectors $\{e_0, e_1, \cdots, e_{n-1}\}$.

Now every point X can be written as:

$$X = \sum_{j=0}^{n-1} x_j e_j.$$

In order to understand how easy it is to take the derivative of X (here we talk about the identity function $f(X) = X$ to be precise) we observe:

$$\frac{\partial X}{\partial x_j} = e_j \text{ or, if you want } \frac{\partial f}{\partial x_j} = e_j.$$

Furthermore the natural basis is spanned by powers of the second basis vector e_1, for $0 \leq j < n$ we have:

$$e_1^j = e_j.$$

With this you automatically get for $i + j < n - 1$:

$$e_i e_j = e_j e_i = e_{i+j}.$$

And so for every two points X and Y you have $XY = YX$ (the symmetry condition is also needed when you want to have derivatives in the lazy way).
Now we have enough ammo to parameterize the multiplication in n dimensions; the parameter is always written as a capital T and it is simply the n-th power of the second basis vector:

$$T := e_1^n = \sum_{j=0}^{n-1} t_j e_j.$$

In the complex plane you can find the derivative of a function rather simple because it does not make any difference from what direction you take the limit for the derivative,in n dimensional space we want the same ease and therefore the real component of the parameter T cannot be zero:

$$t_0 \neq 0.$$

Another way of understanding this 'technical condition' that t_0 cannot be zero is by observing that allowing it to be zero, you only have some direct sum of two spaces. The complex plane does not work that way and in n dimensional space it does not work that way. (The technical condition guarantees that e_1, and thus all it's powers, has an inverse.)

To sum it up: We use the standard natural basis with unit basis vectors, multiplications are parameterized by T (of course T is only a point in our n dimensional space) and in order to have the easy life we need $t_0 \neq 0$.

Generalizing the Cauchy Riemann equations. In the theory of functions on the complex plane, a function has to satisfy the so called Cauchy Riemann equations in order to be an analytic function.
There are two ways of writing down the Cauchy Riemann equations, a clumsy one and an elegant one.

Lets first do it clumsy:
A complex number z is written in real numbers like $z = x + iy$ and so a function from the complex plane to itself can be written as:

$$f(x, y) = u(x, y) + iv(x, y)$$

In that case f is analytic if:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

If you see these for the first time in your life, rather likely you need over 10 seconds to grasp what is going on.

But it is very easy to see what is going on if you write it like:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} i$$

Simply because

$$\frac{\partial z}{\partial x} = 1, \text{ and } \frac{\partial z}{\partial y} = i.$$

So the Cauchy Riemann equations are very simple if you write them down in an easy to understand way (with already the basics of the chain rule for differentiation in it).
For a function f from n dimensional space to itself this translates to:

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_0} e_i.$$

And these are the generalized version of the Cauchy Riemann equations.
From the above you can find the relations between the diverse components

$$\frac{\partial f_i}{\partial x_j},$$

since there are n^2 of these it does not bring much insight.
While simply multiply the derivative by e_i is a very simple thing to do...

Hence it makes sense to write the derivative $f'(X)$ as follows:

$$f'(X) = \frac{\partial f}{\partial X} \overset{\text{def}}{:=} \frac{\partial f}{\partial x_0}.$$

To put it simple: You can find the derivative via differentiating into any kind of allowed direction, but most simple is just to differentiate into the direction of the real axis (the direction of the real numbers).

You might think there could be a problem if you have found a vector direction (a point) that is not invertible. Because if you try to calculate the derivative via:

$$\lim_{X \to A} \frac{f(X) - f(A)}{X - A},$$

could there be a problem if you cannot divide by $X - A$?
There is no problem at all, the combined collection of non invertible elements has always lesser dimension than n and as such is thin in n dimensional space. If you have found a non invertible element, say N, in that case all real multiples are also non invertible so you can calculate a unit vector in that direction, say e_N.

If you want to know the derivative in the N direction at a point X you simply calculate $f'(X)e_N$...
(The collection of non invertible points form an ideal in the n dimensional space under multiplication.)

The obvious relation with matrix multiplication. Once you have found your parameter T for a multiplication, it is very easy to make a matrix representation for this. Before we do that lets look at two equivalent ways to view the multiplication of an nxn matrix with a column matrix (a point X written vertical). You can view a matrix as rows stacked upon each other:

$$M = \begin{pmatrix} R_0 \\ R_1 \\ \vdots \\ R_{n-1} \end{pmatrix} \text{ and } MX = \begin{pmatrix} < R_0, X > \\ < R_1, X > \\ \vdots \\ < R_{n-1}, X > \end{pmatrix}$$

In this $< R_0, X >$ means of course the standard inner product.
If X is one of the basis vectors e_i you will get the $i+1$ column of the matrix.
Equivalent a matrix is a bunch of columns next to each other:

$$M = \begin{pmatrix} C_0 & C_1 & \dots & C_{n-1} \end{pmatrix} \text{ so } MX = \begin{pmatrix} C_0 & C_1 & \dots & C_{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \sum_{i=0}^{n-1} C_i x_i.$$

The above is so elementary, it is in every first year of university or similar education. It is almost an insult to the reader to even post it. Therefore we now craft the matrix representation $M(X)$ of a point X. That is also very easy:

$$M(X) = \begin{pmatrix} X & Xe_1 & Xe_2 & \dots & Xe_{n-1} \end{pmatrix}$$

I would like to make no difference between a point X written as $(x_0, \dots, x_{n-1})$ or as a column matrix in order to avoid constantly taking transposes that make the text hard to read. So $M(X)$ contains columns, for example the second column can be calculated as next:

$$Xe_1 = \begin{pmatrix} 0 \\ x_0 \\ x_1 \\ \vdots \\ x_{n-2} \end{pmatrix} + x_{n-1}T.$$

Furthermore once you have a matrix representation you can go back via $M(X)e_0$ because X is always the first column of the matrix representation.
Because most math computer programs have ready to use matrix algorithms this saves a lot of work.

More things that are utterly obvious:
$M(XY) = M(X)M(Y)$ and X has an inverse if $\det(MX)) \neq 0$ and you can use your computer program to calculate the inverse via:

$$X^{-1} = M(X)^{-1}e_0.$$

I am sorry I have to insult you further, but it is also obvious that $M(e_0) = 1$ (the unit matrix) and that

$$M(e_1) = \begin{pmatrix} e_1 & e_2 & \dots & e_{n-1} & T \end{pmatrix}$$

A simple example in the plane, take $T = -0.6 + 0.4e_1$. So now our parameter $e_1^2 = T$ is as above, remark I have chosen T of unit length.
The matrix representation of a point $X = x_0 + x_1e_1$ now becomes:

$$M(X) = \begin{pmatrix} x_0 & x_1t_0 \\ x_1 & x_0 + x_1t_1 \end{pmatrix}$$

The determinant becomes $\det(M(X)) = x_0^2 + t_1x_0x_1 - t_0x_1^2$ and in our simple example this becomes:
$\det(M(X)) = x_0^2 + 0.4x_0x_1 + 0.6x_1^2$.
It is not hard to show the determinant never vanished when $X \neq 0$, I think (or I hope) that the next method gives the least work:
A non invertible point is always on a linear subset thus on a line through zero.
Just substitute $x_0 = 1$ and proof it cannot be solved;
Substitute $x_1 = 1$ and do the same.
Of course remark that both lines $x_0 = 1$ and $x_1 = 1$ coincidence with all lines through the origin (expect for both x_0 and x_1 axis who were non invertible by definition).

With this parameter T you can also solve $X^2 + 1 = 0$, if I made no calculation error the solution is given by $x_0 = \sqrt{1/14}$ and $x_1 = -5\sqrt{1/14}$.

Exercise: Check if indeed I did not make a calculation error...

It is now also easy to give an isomorphism between the complex plane and our plane with parameter T, since in the complex plane the solution of $z^2 = -1$ is usually written as i our isomorphism takes the from:

$$\phi(1) = e_0 (= 1 \text{ of course}) \text{ and } \phi(i) = \sqrt{1/14} - 5\sqrt{1/14}e_1.$$

Now we have our isomorphism (I did not proof $\phi(z_1 + z_2) = \phi(z_1) + \phi(z_2)$ and $\phi(z_1z_2) = \phi(z_1)\phi(z_2)$!) we can borrow the norm from the complex plane via constructing the inverse of ϕ and say $|X| := ||\phi^{-1}(X)||$. Where of course $||z|| = ||x + iy|| = \sqrt{x^2 + y^2}$.

Newton mechanics: Velocity and force fields, kinetic energy (weird stuff). This paragraph contains weird mechanics because (according to my humble opinion) it does not make much sense to study liquid floods (or gasses or plasma) in lets say a 10 dimensional space. I consider this mathematical fantasies but the results are rather nice so why not write them down?
Lets do it in Newton's dot notation (the dot only means differentiation to time).

$$\dot{X} = f(X).$$

This simply means the next: at every point X we hang a velocity vector $f(X)$ where f is any given analytical function.What is in that case the acceleration? Well:

$$\ddot{X} = \frac{\partial}{\partial t} f(X) = \frac{\partial f}{\partial X}\frac{\partial X}{\partial t} = f'(X)\ \dot{X} = f'(X)f(X).$$

Thus:

$$\ddot{X} = \frac{\partial}{\partial X}\frac{1}{2} f(X)^2.$$

If we write v and a for velocity and acceleration and multiply left and right with a constant m we get:

$$ma = \frac{\partial}{\partial X}\frac{1}{2} mv^2.$$

At last writing $F = ma$ and $E_k = \frac{1}{2}mv^2$ this becomes:

$$F = \frac{\partial}{\partial X} E_k.$$

In words this would be:
The force field is the spatial derivative of the kinetic energy.