Clifford algebras past and present or, square style versus cubic style.

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Introduction. Originally Clifford algebras were a way to more or less unify things like the complex plane $\mathbb{C}$ and the quaternions $\mathbb{H}$ from sir Hamilton.

In general the elements of Clifford algebras do not commute, therefore they are not suitable for complex spaces like $\mathbb{R}^n$ with $n > 2$.
There are a whole lot of definitions out there but most writers define Clifford algebras rather clumsy and highly counter intuitive to upright confusing.
So with two relatively simple Clifford algebras build on top of $\mathbb{R}^3$ and $\mathbb{R}^4$ I will try to give a simple way of understanding all those vague definitions...

The old school Clifford algebras are always strongly related to the square; the square of any basis vector is either +1 or -1.
In the main part of this update I will define Clifford algebras 'cube style' where as you might expect the cube of the basis vectors is always +1 or -1.

Clarification of notation: For basis vectors we will use e_0, e_1, e_2 and so on and so on.
There are also bivectors, trivectors etc; bivectors are denoted as e_{12} meaning

$$e_{12} = e_1 e_2$$

and trivectors are denoted like e_{123} meaning

$$e_{123} = e_1 e_2 e_3.$$

I will use these notations for square old school Clifford algebras and the cube style Clifford algebras.
End of the **clarification of notation** and also **end of the introduction.**

Simple example 1: The complex plane $\mathbb{C}$.
The complex plane has two basis vectors

$$e_0 = 1 \text{ and } e_1 = i$$

and since $e_1^2 = i^2 = -1$ there is no Clifford algebra possible beyond these two dimensions.
Simple example 2: The complex space $\mathbb{R}^3$.
This complex space has three basis vectors

$$e_0 = 1, e_1 = j \text{ and } e_2 = j^2$$

and since $e_1 e_2 = j \cdot j^2 = j^3 = -1$ for the complex multiplication, also it is not possible to enlarge this three dimensional space any further.
These two examples might look very simple but all complex and circulant number systems in whatever dimension are always closed spaces under addition and multiplication...

That brings us to the very first example that is a little bit more complicated: the Clifford algebra on top of $\mathbb{R}^3$:
Example 3 As usual the Clifford algebra's also use the orthogonal basis vectors of unit length. So the three basis vectors of $\mathbb{R}^3$ are given by

$$\begin{aligned} 1 = \ e_0 &= (1,0,0) \\ e_1 &= (0,1,0) \\ e_2 &= (0,0,1) \end{aligned}$$

And as far as I understand the old school Clifford algebras, the squares are all 1:

$$\begin{aligned} e_0^2 &= 1 \\ e_1^2 &= 1 \\ e_2^2 &= 1 \end{aligned}$$

But now there is one bivector possible and that is $e_{12} = e_1 e_2$ and with Clifford algebras you always have that they anti-commute if you have two different one-vectors like e_1 and e_2. So

$$\begin{aligned} e_1 e_2 &= -e_2 e_1 \quad \text{or} \\ e_{12} &= -e_{21} \end{aligned}$$

Armed with this anti-commute knowledge it is possible to calculate the square of e_{12}, the result is

$$e_{12}^2 = e_1 e_2 e_1 e_2 = -e_1 e_1 e_2 e_2 = -1$$

Because of this behaviour the pair of basis vectors $\{e_0, e_{12}\}$ is the same as the complex plane $\mathbb{C}$ that has the basis written as $\{1, i\}$ with $i^2 = -1$.

The Clifford algebra on top of $\mathbb{R}^3$ is a four dimensional thing and the basis is given by

$$\{e_0, e_1, e_2, e_{12}\}$$

It is **important to remark** that this 4D space is closed under addition and multiplication. I am not going to write down the entire multiplication table but it is obvious that for example

$$e_1 e_{12} = e_2 \text{ and } e_{12} e_1 = -e_2$$
$$e_{12} e_2 = e_1 \text{ and } e_2 e_{12} = -e_1$$

You might wonder why Mr. Clifford used the anti-community stuff because with that you throw all things related to easy differentiation over board? Very likely Clifford wanted to hook on to the quaternions from sir Hamilton, of course I can not prove this statement but most of the time scientists try to improve on the results that are already known...
And the quaternions are loaded with anti-commuting stuff; the three imaginary components i, j and k all anti commute like $ji = -ij$ etc etc.

Example 4: The Clifford algebra on top of $\mathbb{R}^4$.
Our starting basis vectors are now given by

$$\begin{aligned} 1 = \ e_0 &= (1, 0, 0, 0) \\ e_1 &= (0, 1, 0, 0) \\ e_2 &= (0, 0, 1, 0) \\ e_3 &= (0, 0, 0, 1) \end{aligned}$$

By definition they all square to $+1$, so

$$e_0^2 = e_1^2 = e_2^2 = e_3^2 = 1$$

Needless to say, this is **extremely boring behaviour** as exposed by the first four basis vectors of this Clifford algebra. But at least now we have a few more of those bi and tri - vectors.
If you understood example number 3, it is not hard to understand why the Clifford algebra on top of $\mathbb{R}^4$ is given by the next set of orthogonal basis vectors:

$$\{e_0, e_1, e_2, e_3, e_{23}, e_{13}, e_{12}, e_{123}\}$$

Where, of course, $e_{123} = e_1e_2e_3$.
There is a lot of perpendicular stuff going on with these 8 basis vectors, for example
$e_{23}e_{123} = -e_1$ where $e_1 \perp e_{23}$ and $e_1 \perp e_{123}$

It is not hard to calculate the squares of the last four basis vectors, they all square to minus one.
As an exercise you can try it for e_{23}, e_{13} and E_{12}.
For reasons of completeness I will calculate the square of the last basis vector, the trivector.
With the notation $\overset{(2)}{=}$ I mean that two swaps are don from going to the left of the '=' sign to the right of the = sign:

$$(e_{123})^2 = \ e_1e_2e_3e_1e_2e_3 \ \overset{(2)}{=} \ {e_1}^2e_2e_3e_2e_3 \ \overset{(1)}{=} \ -1$$

So far the examples from the Clifford algebras **square style**, we proceed how such algebras would look if we use **cubic style** as this idea is borrowed from the 3D complex numbers.

In the **simple example 2** above we observed that it is not possible to extend $\mathbb{R}^3$ if the basis vectors already give rise to a circular or complex 3D number system.
Because if the basis of $\mathbb{R}^3$ is denoted as $\{e_0, e_1, e_3\}$, in that case
$e_{12} = +1$ for the circular multiplication and
$e_{12} = -1$for the complex version.
So extension of $\mathbb{R}^3$ is not possible because it is already a closed space under addition and multiplication.

Now I construct a five dimensional space, basically two versions of circular numbers from $\mathbb{R}^3$ but the real axis is used jointly.
Our starting space $\mathbb{R}^5$ now has a basis like

$$\{e_0, e_1, e_{11}, e_2, e_{22}\} \text{ where } e_{11} = {e_1}^2 \text{ and } e_{22} = {e_2}^2.$$

This is similar to as we always write $\{1, j, j^2\}$ for the 3D complex & circular number spaces.
The third powers of these five basis vectors all equal 1 (of course e_0 is already 1).
On top of this $\mathbb{R}^5$ there is one extra bivector $e_{12} = e_1e_2$. There are two trivectors

$e_{122} = e_1 e_{22}$ and $e_{211} = e_2 e_{11}$.
At last we have one quadvector $e_{1122} = e_{11} e_{22}$.
On top of our $\mathbb{R}^5$ we now have the cubic
Clifford algebra with basis

$$\{e_0, e_1, e_{11}, e_2, e_{22}, e_{12}, e_{122}, e_{211}, e_{1122}\}$$

Of course e_{111} is not an extra basis vector because it is 1,
the same goes for e_{222}. Since this Clifford algebra
has nine basis vectors, it is a $\mathbb{R}^9$ where the
first five basis vectors came from our gluing together
two versions of $\mathbb{R}^3$ along the real axis.

For Clifford algebras **square style** the square of
all basis vectors equals plus or minus one.
Here we will check how the extra four basis vectors
behave if we raise them to the third power.

Again with the notation $\overset{(3)}{=}$ it is the use of
three swaps to proceed in the calculation.
Basis vector number six ${e_{12}}^3$:

$$\begin{aligned} {e_{12}}^3 &= e_1 e_2 e_1 e_2 e_1 e_2 \\ &\overset{(1)}{=} -{e_1}^2 {e_2}^2 e_1 e_2 \\ &\overset{(2)}{=} -{e_1}^3 {e_2}^3 = \\ &= -1 \end{aligned}$$

Before we proceed with e_{122} recall that e_{22}
is the square of e_2 and therefore, for example,
$e_{22} e_1 = e_1 e_{22}$ because you need two swaps
in order to get e_1 from the right to the left...

Basis vector number seven ${e_{122}}^3$:

$$\begin{aligned} {e_{122}}^3 &= e_1 e_{22} e_1 e_{22} e_1 e_{22} \\ &\overset{(2)}{=} {e_1}^2 {e_{22}}^2 e_1 e_{22} \\ &\overset{(4)}{=} {e_1}^3 {e_{22}}^3 = \\ &= 1 \end{aligned}$$

Very likely the next basis vector behaves the same,
let's check it out.
Basis vector number eight ${e_{211}}^3$:

$$\begin{aligned} {e_{211}}^3 &= e_2 e_{11} e_2 e_{11} e_2 e_{11} \\ &\overset{(2)}{=} {e_2}^2 {e_{11}}^2 e_2 e_{11} \\ &\overset{(4)}{=} {e_2}^3 {e_{11}}^3 = \\ &= 1 \end{aligned}$$

So what will happen if we raise our last basis vector e_{1122} to the third power?
Let's check it out.
Basis vector number nine ${e_{1122}}^3$:

$$\begin{aligned} {e_{1122}}^3 &= e_{11}e_{22}e_{11}e_{22}e_{11}e_{22} \\ &\overset{(4)}{=} {e_{11}}^2{e_{22}}^2e_{11}e_{22} \\ &\overset{(8)}{=} {e_{11}}^3{e_{22}}^3 = \\ &= 1 \end{aligned}$$

So one conclusion is pretty clear: The behaviour of cubic Clifford algebras is completely different from the behaviour of the square style Clifford algebras.

But with that conclusion there is nothing new under the sun, the 3D complex numbers also behave very different if you compare them to the 2D complex numbers from the complex plane $\mathbb{C}$.