The story of π and the radius, or: how to square the circle properly.

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1 Introduction.

A long long time ago when my readers were in highschool or at the university they were likely confronted by evildoing math teachers who forced them in understanding formulas like the next upon a circle:

$$L = 2\pi r, A = \pi \cdot r^2$$

Where L and A stand for the length of the boudery and area of a circle disk with radius r. May be deep down in the caves of the mind of my readers they even remember their evil math teachers saying that L was the derivate of the area A.

Without doubt these were horrible days for most of my readers and those rare ones who even liked torture like that wisely kept their mouth shut to aviod being 'Guantonama Bayed' by their fellows.
Those were the days of growing pubic hair and 'how do I look' kind of anxiety. And on top of that the evildoing math teachers made life only more difficult with stating that the volume of a sphere with radius r was given by:

$$V = \frac{4}{3}\pi \cdot r^3$$

Sheer horror of course and to most of you completely useless knowledge, that was a fact of life in those days before you understood the Black-Scholes model for pricing derivates.

Yet without mercy those torturing math teachers from hell went on with the area of a sphere while you tried to look inside the blouse of Mary Jane:

$$A = 4\pi \cdot r^2.$$

May be the shape of Mary Jane is stored somewhere else in the caves of your brain, but also the area A could be remembered for use during exams as being the derivate of the volume:

$$A = \frac{dV}{dr} \text{ or } A = V'$$

2 A few examples.

Example 1: A square with side (edge) a has an area of $A = a^2$ while the length of the four sides combine to $L = 4a$. We see $L \neq A'$, but with observing that a 'radius' r like $a = 2r$ nicely fits we get:

$$A = 4r^2 \text{ and } L = 8r \text{ so } L = \frac{\partial A}{\partial r}.$$

Example 2: A cube with side a has a volume of $V = a^3$ and a total area of $A = 6a^2$, again it 'does not fit' the $A = V'$ equation. But again with choosing $a = 2r$ we get:

$$V = 8r^3 \text{ and } A = 24r^2 \text{ so } A = \frac{\partial V}{\partial r}.$$

Example 3: A four dimensional hypercube with side a has a four dimensional volume $V_4 = a^4$ and a three dimensional 'area' of $V_3 = 8a^3$.
Now it is getting boring but again with taking $a = 2r$ we get:

$$V_4 = 16r^4 \text{ and } V_3 = 64r^3 \text{ so } V_3 = \frac{\partial V_4}{\partial r}.$$

Example 4: The n-dimensional cube will have in a similar fashion:
$V_n = 2^n r^n$ and an 'area' of $V_{n-1} = 2n \cdot 2^{n-1} r^{n-1} = n 2^n r^{n-1}$ so boring boring we have:

$$V_{n-1} = \frac{\partial V_n}{\partial r}.$$

Example 5: An equilateral triangle $\triangle ABC$ has obvious a total lenght and area of

$$L = 3a \text{ and area } A = \frac{1}{4}\sqrt{3}a^2$$

When we put $a = 2\sqrt{3}r$ we get the next:

$$A = \frac{1}{4}\sqrt{3}a^2 = 3\sqrt{3}r^2 \text{ and } L = 3a = 6\sqrt{3}r \text{ so } L = \frac{\partial A}{\partial r}.$$

2.1 What have these five examples in common?

They have in common you can write 'volume' V_n and 'area' V_{n-1} (in the n-dimensional sense) as follows:

$$V_n = c \cdot r^n \text{ and } V_{n-1} = nc \cdot r^{n-1}.$$

Once you have the V_n and V_{n-1} represented as numbers or depending on other parameters describing the geometrical object you can easily calculate it's $'\pi'$ value c and it's 'radius' r via:

$$r = \frac{n \cdot V_n}{V_{n-1}}$$

$$c = \frac{V_{n-1}^n}{n^n \cdot V_n^{n-1}}$$

2.2 Exercise, triangles.

Draw just a triangle $\triangle ABC$ and construct the inscribed circle in it. Or start with a circle and draw a triangle around it.
Draw three lines from the midpoint of the circle to the corners A, B and C. The angle α at A will get devided in α_1 and α_2, analog at point B and C. Name them α_1 through α_6

Proof that the above mentioned constant c is given by:

$$2c = \sum_{i=1}^{6} \frac{1}{\tan(\alpha_i)}$$

Remark: You can do the same with a square and sum $\sum_{i=1}^{8} 1/\tan(\alpha_i)$ with all $\alpha_i = \pi/4$ to get the $c = 4$ as desired for consistency. End of remark.

2.3 More examples.

Example 6: The tetrahedron, or using the translated term from Dutch, the regular four plane. The regular four plane is made of four regular triangles, when they have side a the area of a triangle is of course $\frac{1}{4}\sqrt{3}a^2$ so area and volume of the regular four plane are:

$$A = \sqrt{3}a^2 \text{ and } V = \frac{1}{12}\sqrt{2}a^3$$

So the radius of the regular four plane is given by:

$$r = 3V/A = \frac{\frac{1}{4}\sqrt{2}a^3}{\sqrt{3}a^2} = \frac{1}{12}\sqrt{6}a$$

And the 'π value' c of the tetrahedron is:

$$c = \frac{A^3}{3^3 V^2} = \frac{(\sqrt{3}a^2)^3}{27 \cdot (\frac{1}{12}\sqrt{2}a^3)^2} = 8\sqrt{3}$$

Example 7: A non example in case the dear reader got overenthousiastic after solving the above exercise because if that is true why not simply differentiate an ellips to it's radius to get the length of an ellips?
Now it has to be remarked that the length of an elliptic shaped curve has driven

good mathematicians to madness in the past, but a simple and interesting observation can be made.

An ellips with two half axis of length a and b has an area of $A = \pi ab$ and when we parametrisize this ellips with $\phi(t) = (a\cos t, b\sin t)$ we find for the length of course $L = \int_0^{2\pi} ||\phi'(t)||dt = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}dt$ therefore when we differentiate the area to it's radius we get:

$$\frac{\partial}{\partial r}\pi ab = \pi\frac{\partial a}{\partial r}b + \pi a\frac{\partial b}{\partial r} = \pi c_1 b + \pi a c_2 = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}dt.$$

This for two constants c_1 and c_2. That is all there is, I personally found the representation via two constants of interest but it does not solve the problem of the length of an ellips in elementary functions.
(A second differentiation $\partial^2 O/\partial r^2$ gives the relation $c = \pi c_1 c_2$ between the diverse constants.)

Example 8: Take a cylinder with radius r and height $2r$, a bit of elementary calculation gives that the 'π value' $c = 2\pi$ and the radius is simply r. We see once more that when a circle or a sphere fits into the object in question the radius is also that of the circle/sphere. Only the constant c is different.

Example 9: Take a cone with circular base of radius R, let C denote the base boundary (a circle). Make the height of the cone $h = \sqrt{3}R$ so the distance from the top to C equals $2R$.
Area and volume expressed in the parameter R are given by $A = 3\pi R^2$ and $V = \frac{1}{3}\sqrt{3}\pi R^3$. So the radius and '$\pi$ value' of this cone are:

$$r = \frac{3V}{A} = \frac{1}{3}\sqrt{3}R \text{ and } c = \frac{A^3}{27V^2} = 3\pi$$

Again remark that r is also the radius of the sphere that fits in this cone. It should be noted too that 'fitting into something' is not a nessecary condition to have the overlap between 'radius of the object' and the radius of the circle/sphere that fits within.

2.4 End.

So far this simple to understand and easy mathematics, I hope you have never seen it before and I hope it comes as a surprise to you. So simple and so little in the textbooks, may I invite writers of first year university courses to include this topic in one or more exercises?

Originally I started writing this down after 11 years to make it some Christmas gift for the hedge funds and money traders, it is now April 2005 and only last week I picked this up again. After the usual problems around the Latex package were solved it is now ready for publication.

Sincerely yours, 11 April 2005, Reinko Venema, Groningen.